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Probability Density Functions of Envelope and Phase of the Sum of a PSK Modulated Carrier and Narrowband Gaussian Noise

Author: Lothar Friederichs, AUDENS Telecommunications Consulting

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with corrections and extensions[1] of 18.07.2008

Contents

[1] The corrections concern only some details in the transition from equation (2-8) to equation (2-9), not the end result. The extensions introduce Complementary Cumulative Distribution Functions as useful applications of the phase PFD.

1. Scope

In many applications it is of interest to have an analytical expression for the Probability Density Functions (PDF) of the envelope and the phase of the sum of an M-ary PSK modulated or an unmodulated sine wave (e.g. $M = 2, 4, 8 \ldots$) and additive narrowband Gaussian noise under the conditions of coherent or non-coherent reception, i.e. including the impact of carrier frequency off-set. The wanted PDF expressions are derived here along the lines shown e.g. in ref. [1] or [2] for the case of an unmodulated carrier plus narrowband Gaussian noise.

As a case of particular interest, the phase density function of an unmodulated carrier plus NB Gaussian noise has been applied to present Complementary Cumulative Distribution Functions of the phase and phase difference.

Grateful acknowledgement is due to Wolfgang Steinert, who stimulated the investigations performed and confirmed the correctness of the analytical results by rigorous simulations run under MATLAB.

2. Unmodulated Carrier plus Narrowband Gaussian Noise [1, 2]

PDF of Amplitude Envelope

The carrier shall have constant, but uniformly distributed phase $\psi \in (0,2\pi)$, independent of the noise process represented by a sample function $x(t) = x_c(t) \cdot \cos\omega_c t - x_s(t) \cdot \sin\omega_c t$.

Here, $x_c(t)$ and $x_s(t)$ are independent Gaussian variables with zero mean and variance σ_x^2. The sum of carrier and noise is

$$y(t) = P\cos[\omega_c t + \psi] + x(t) = X_c(t) \cdot \cos\omega_c t - X_s(t) \cdot \sin\omega_c t \qquad (2\text{-}1)$$

or

$$\begin{aligned} y(t) &= V(t) \cdot \cos[\omega_c t + \psi + \phi(t)] \\ &= V(t) \cdot \cos[\psi + \phi(t)] \cdot \cos\omega_c t - V(t) \cdot \sin[\psi + \phi(t)] \cdot \sin\omega_c t \end{aligned} \qquad (2\text{-}2)$$

with

$$X_c(t) = P\cos\psi + x_c(t) = V(t) \cdot \cos[\psi + \phi(t)] \qquad (2\text{-}3a)$$

and

$$X_s(t) = P\sin\psi + x_s(t) = V(t) \cdot \sin[\psi + \phi(t)]. \qquad (2\text{-}3b)$$

Since $x_c(t)$, $x_s(t)$ and ψ are stastically independent, the joint probability density function of X_{ct} representing the possible values of $X_c(t)$, X_{st} representing the possible values of $X_s(t)$ and ψ can be factorised into the single PDFs of X_{ct}, X_{st} and ψ.

The triplet of random variables X_{ct}, X_{st} and ψ is transformed by equation (2-3) into the triplet of random variables V_t, ϕ_t and ψ where V_t and ϕ_t represent the possible values of $V(t)$ and $\phi(t)$. The transformation determinant ("Jacobian") is

$$J = \begin{vmatrix} \partial X_{ct}/\partial V_t & \partial X_{st}/\partial V_t & \partial\psi/\partial V_t \\ \partial X_{ct}/\partial\phi_t & \partial X_{st}/\partial\phi_t & \partial\psi/\partial\phi_t \\ \partial X_{ct}/\partial\psi & \partial X_{st}/\partial\psi & \partial\psi/\partial\psi \end{vmatrix} = \begin{vmatrix} \cos[\psi+\phi_t] & \sin[\psi+\phi_t] & 0 \\ -V_t\sin[\psi+\phi_t] & V_t\cos[\psi+\phi_t] & 0 \\ -V_t\sin[\psi+\phi_t] & V_t\cos[\psi+\phi_t] & 1 \end{vmatrix} = V_t.$$

The joint probability density function of V_t, ϕ_t and ψ is now

$$p(V_t,\phi_t,\psi) = p(X_{ct},X_{st},\psi)\cdot|V_t| = p(X_{ct})\cdot p(X_{st})\cdot p(\psi)\cdot|V_t|$$
$$= \frac{1}{4\pi^2\sigma_x^2}\cdot\exp\left[-\frac{1}{2\sigma_x^2}\left\{\begin{matrix} V_t^2\cos^2(\psi+\phi_t)+V_t^2\sin^2(\psi+\phi_t)+P^2 \\ -2PV_t[\cos(\psi+\phi_t)\cos\psi+\sin(\psi+\phi_t)\sin\psi]\end{matrix}\right\}\right]\cdot|V_t|$$

$$p(V_t,\phi_t,\psi) = \frac{V_t}{4\pi^2\sigma_x^2}\cdot\exp\left[-\frac{P^2\sin^2\phi_t}{2\sigma_x^2}\right]\cdot\exp\left[-\frac{(V_t-P\cos\phi_t)^2}{2\sigma_x^2}\right] \quad ;V_t\geq 0 \qquad (2\text{-}4)$$

By integration of equ. (2-4) over ψ the joint probability density function of instananeous amplitude (or envelope) V_t and phase ϕ_t is obtained.

$$p(V_t,\phi_t) = \frac{V_t}{2\pi\sigma_x^2}\cdot\exp\left[-\frac{P^2\sin^2\phi_t}{2\sigma_x^2}\right]\cdot\exp\left[-\frac{(V_t-P\cos\phi_t)^2}{2\sigma_x^2}\right] \quad ;\ V_t\geq 0 \qquad (2\text{-}5)$$

The PDF of the envelope of the sum of an unmodulated carrier and narrowband Gaussian noise is now determined by integrating equ. (2-5) with respect to ϕ_t:

$$p(V_t) = \frac{V_t}{2\pi\sigma_x^2}\int_0^{2\pi}\exp\left[-\frac{1}{2\sigma_x^2}\left\{V_t^2-2V_tP\cos\phi_t+P^2\right\}\right]d\phi_t$$
$$= \frac{V_t}{2\pi\sigma_x^2}\exp\left[-\frac{V_t^2+P^2}{2\sigma_x^2}\right]\int_0^{2\pi}\exp\left[\frac{V_tP}{\sigma_x^2}\cos\phi_t\right]d\phi_t \qquad (2\text{-}6)$$

Using the Jacobi-Anger relation

$$\exp(z\cos\theta) = \sum_{n=0}^{\infty}\varepsilon_n\cdot I_n(z)\cdot\cos n\theta \qquad \text{where}\quad \varepsilon_n = \begin{cases}1 & \text{for } n=0\\ 2 & \text{for } n>0\end{cases}$$

where I_n denotes the modified Bessel function of first kind, the integral in equ. (2-6) becomes

$$\int_0^{2\pi}\exp\left[\frac{V_tP}{\sigma_x^2}\cos\phi_t\right]d\phi_t = \int_0^{2\pi}I_0\left(\frac{V_tP}{\sigma_x^2}\right)d\phi_t + 2\sum_{n=1}^{\infty}I_n\left(\frac{V_tP}{\sigma_x^2}\right)\int_0^{2\pi}\cos n\phi_t\ d\phi_t\ .$$

It is obvious that the integrals on the right-hand side for $n\geq 1$ will vanish. Thus, the PDF of the envelope of the sum of an unmodulated carrier and narrowband Gaussian noise is

$$p(V_t) = \frac{V_t}{\sigma_x^2}\exp\left[-\frac{V_t^2+P^2}{2\sigma_x^2}\right]\cdot I_0\left(\frac{V_tP}{\sigma_x^2}\right) \qquad (2\text{-}7)$$

Equ. (2-7) describes a Rice probability density function. It is illustrated for various C/N ratios in Fig. 2-1 with normalisation to carrier amplitude and in Fig. 2-2 with normalisation to the noise standard deviation. For $P \rightarrow 0$ (i.e. pure noise), the Rice PDF approaches the Rayleigh PDF.

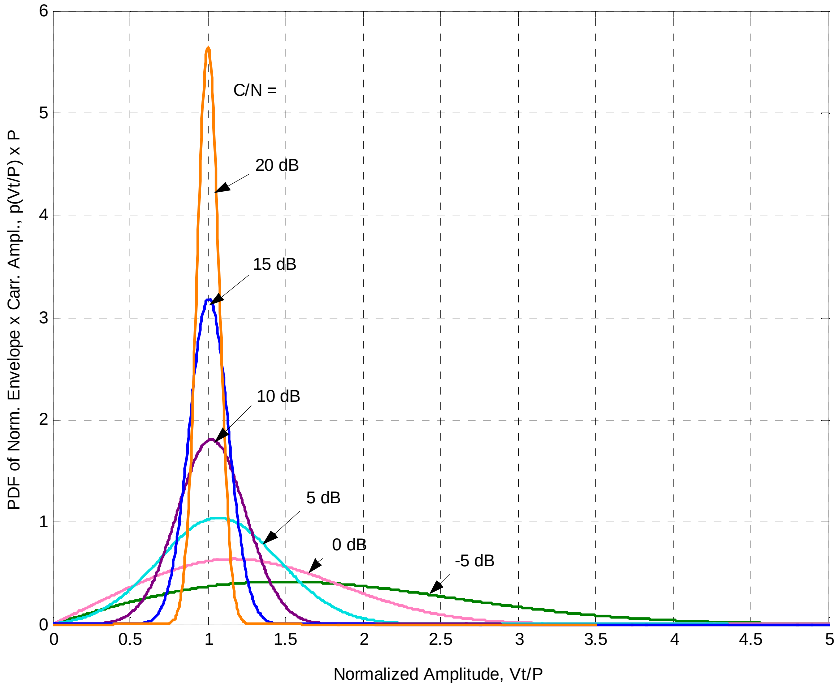

Fig. 2-1: PDF of the Envelope of an Unmodulated Carrier plus Gaussian Noise (Normalisation to Carrier Amplitude)

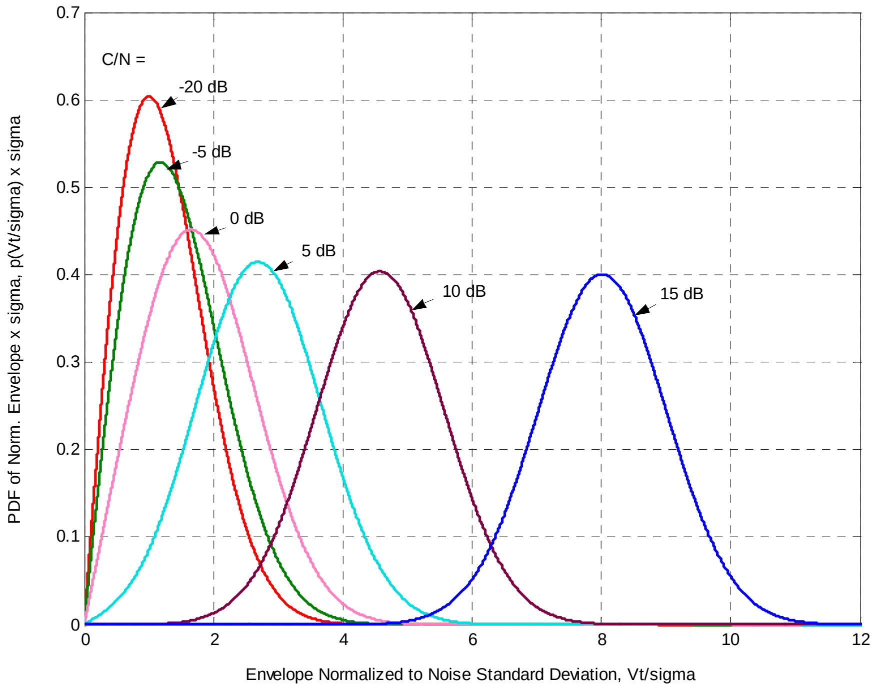

Fig. 2-2: PDF of the Envelope of an Unmodulated Carrier plus Gaussian Noise (Normalisation to Noise Standard Deviation)

PDF of the Phase

Similarly, the PDF of the phase of the sum of an unmodulated carrier and narrowband Gaussian noise is determined by integrating equ. (2-5) with respect to V_t:

$$p(\phi_t) = \frac{1}{2\pi\sigma_x^2} \cdot \exp\left[-\frac{P^2 \sin^2 \phi_t}{2\sigma_x^2}\right] \cdot \int_0^\infty V_t \exp\left[-\frac{(V_t - P\cos\phi_t)^2}{2\sigma_x^2}\right] dV_t \qquad (2\text{-}8)$$

The integral on the right-hand side in equ. (2-8) is solved by substituting $u = \frac{V_t - P\cos\phi_t}{\sqrt{2}\,\sigma_x}$.

The integral then becomes:

$$\int_0^\infty V_t \exp\left[-\frac{(V_t - P\cos\phi_t)^2}{2\sigma_x^2}\right] dV_t = 2\sigma_x^2 \cdot \int_{-P\cos\phi_t/(\sqrt{2}\sigma_x)}^\infty [u + P\cos\phi_t/(\sqrt{2}\sigma_x)] \cdot \exp\left[-u^2\right] du$$

$$= \sigma_x^2 \left[\frac{P\cos\phi_t}{\sqrt{2}\,\sigma_x}\sqrt{\pi} \cdot \mathrm{erfc}\left[-\frac{P\cos\phi_t}{\sqrt{2}\,\sigma_x}\right] + \exp\left[\frac{P^2\cos^2\phi_t}{2\sigma_x^2}\right]\right]$$

The PFD of the phase of an unmodulated carrier and narrowband Gaussian noise for a given ratio $a = P/(\sqrt{2}\ \sigma_x)$ is now:

$$p(\phi_t, a) = \frac{1}{2\pi}\left\{\exp(-a^2) + \sqrt{\pi} \cdot a \cdot \cos\phi_t \cdot \exp\left[-a^2 \cdot \sin^2\phi_t\right] \cdot \mathrm{erfc}\left[-a \cdot \cos\phi_t\right]\right\}, -\pi \le \phi_t \le \pi \qquad (2\text{-}9)$$

Fig. 2-3 gives an illustration of $p(\phi_t, a)$ for various ratios of $C/N = 20\log a$.

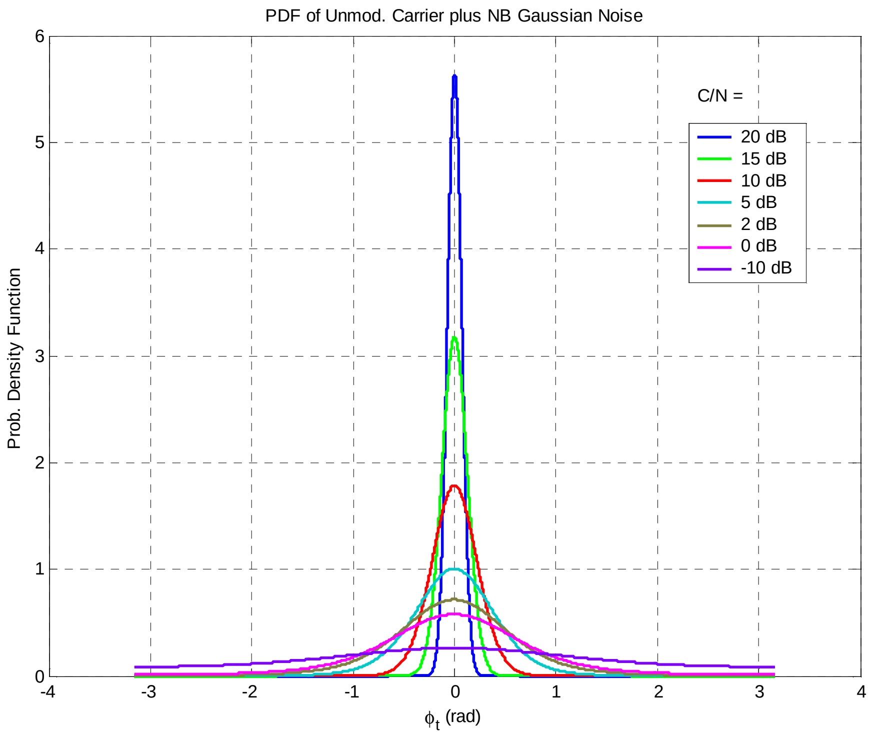

Fig. 2-3: PDF of the Phase of an Unmodulated Carrier plus Narrowband Gaussian Noise

Two limiting cases are of interest:

- Gaussian noise only: $\lim_{P \to 0} p(\phi_t) = \frac{1}{2\pi}$; $-\pi \le \phi_t \le \pi$

  $P \to 0$ is equivalent to $C/N \to -\infty$ dB and the probability density gets rectangular.

- Carrier without noise: $\lim_{\sigma_x \to 0} p(\phi_t) = \delta(0) = \begin{cases} \infty & \text{for } \phi_t = 0 \\ 0 & \text{elsewhere} \end{cases}$ with $\int_{-\pi}^{\pi} \lim_{\sigma_x \to 0} p(\phi_t)\, d\phi_t = 1$

  The probability density reduces to a single line.

Complementary Cumulative Probability Distribution Function

It is of some interest to look at the Complementary Cumulative Probability Distribution Function (CCDF) of the phase of an unmodulated carrier plus narrowband Gaussian noise. The CCDF is obtained from

$$CCDF_p(\varphi, a) = 1 - \int_{-\pi}^{\varphi} p(\phi_t, a)\, d\phi_t \quad \text{with } p(\phi_t, a) \text{ from equ. (2-9).}$$

The integral cannot be solved in closed form. Therefore, it has been evaluated numerically for a set of C/N values. Fig. 2-4a shows the resulting CCDF curves. While the representation in linear scale for the probability seems to take a quite normal shape for φ values close to π, the specific characteristics become evident from the logarithmic representation in Fig. 2-4b. Thus, the CCDF is well suited to reveal the occurrence of extreme values close to π.

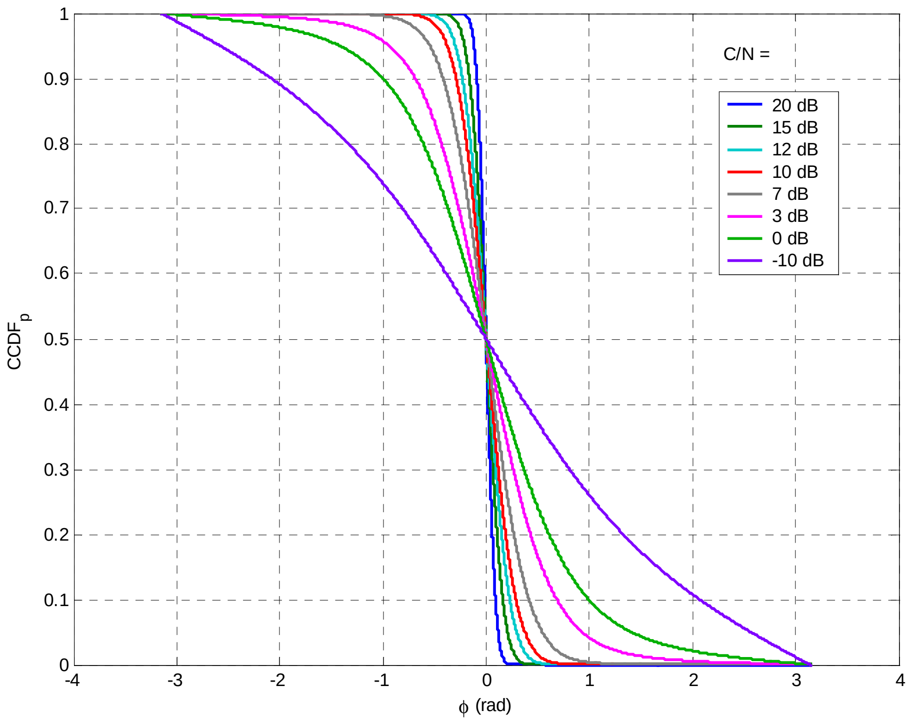

Fig. 2-4a: CCDF of the Phase of an Unmodulated Carrier plus Narrowband Gaussian Noise (Linear Scale)

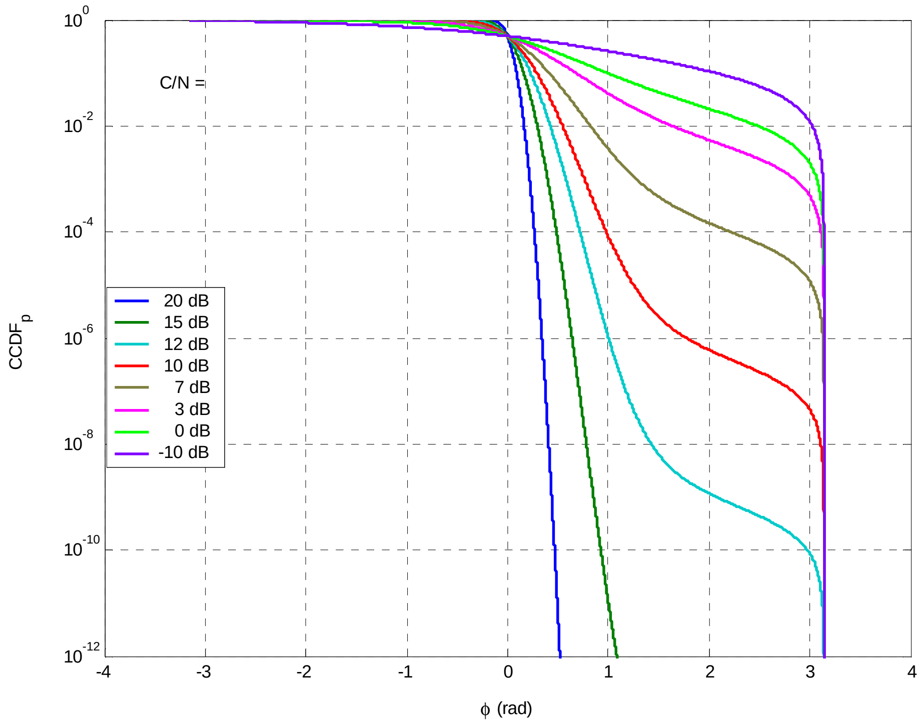

Fig. 2-4b: CCDF of the Phase of an Unmodulated Carrier plus Narrowband Gaussian Noise (Logarithmic Scale)

CCDF of Phase Difference

The relation for the phase of an unmodulated carrier plus Gaussian noise is particularly useful in the analysis of pilot assisted frequency synchronisation techniques based on phase estimation. Here, the distribution of the absolute value of the phase difference Φ of two phasors is of importance. Since the two phasors are statistically independent, the PDF of the difference of their phases results from the convolution of the probability density function according to equ. (2-9) with itself:

$$p_D(\Phi.a) = \int_{-\pi}^{\pi} p(\Phi,a)\cdot p(\phi_t-\Phi,a)\,d\phi_t\ ,\ \text{ where }\quad -\pi\le\Phi\le\pi. \qquad (2\text{-}10)$$

The two-sided CCDF of the phase difference

$$CCDF_{D,2}(\Psi,a) = 1-\int_{-\pi}^{\Psi} p_D(\Phi,a)\ d\Phi\ ,\quad \text{where}\quad -\pi\le\Psi\le\pi\ . \qquad (2\text{-}11)$$

The one-sided CCDF of the absolute value of the phase difference is now

$$CCDF_{D,1}(|\Psi|,a) = 1-2\int_{0}^{|\Psi|} p_D(\Phi,a)\,d\Phi \quad \text{with}\qquad 0\le|\Psi|\le\pi\ . \qquad (2\text{-}12)$$

The results of numerical evaluations of equ. (2-12) are shown in Fig. 2-5a in linear scale and in Fig. 2-5b in logarithmic scale for the same set of C/N values as above. Again, the logarithmic scale in Fig. 2-5b reveals the occurrence of extreme values close to π with rather high probability.

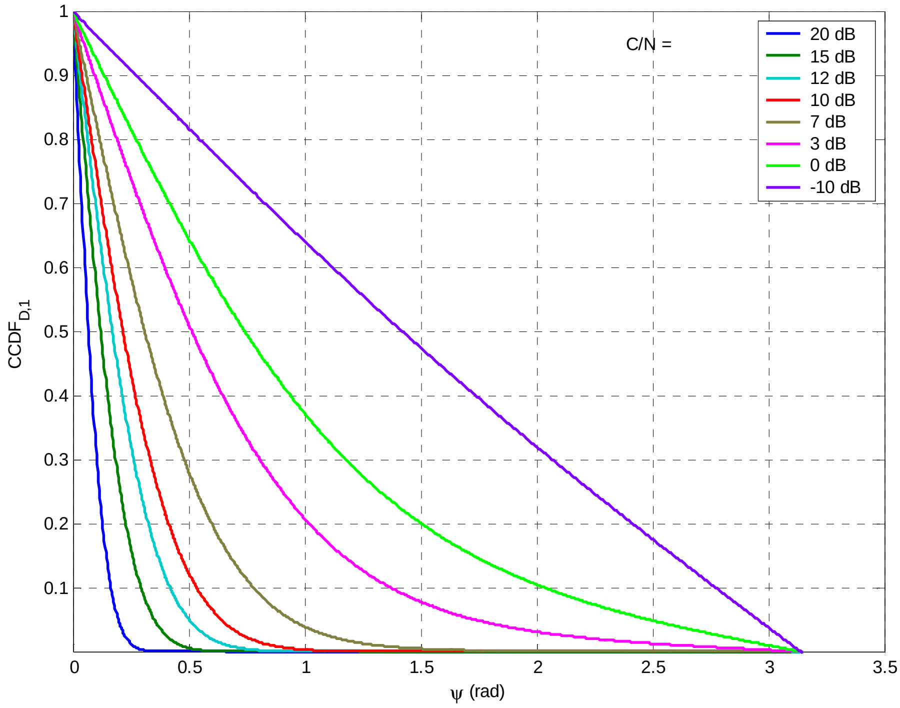

Fig. 2-5a: CCDF of the Phase Difference of two Phasors Representing an Unmodulated Carrier plus Narrowband Gaussian Noise (Linear Scale)

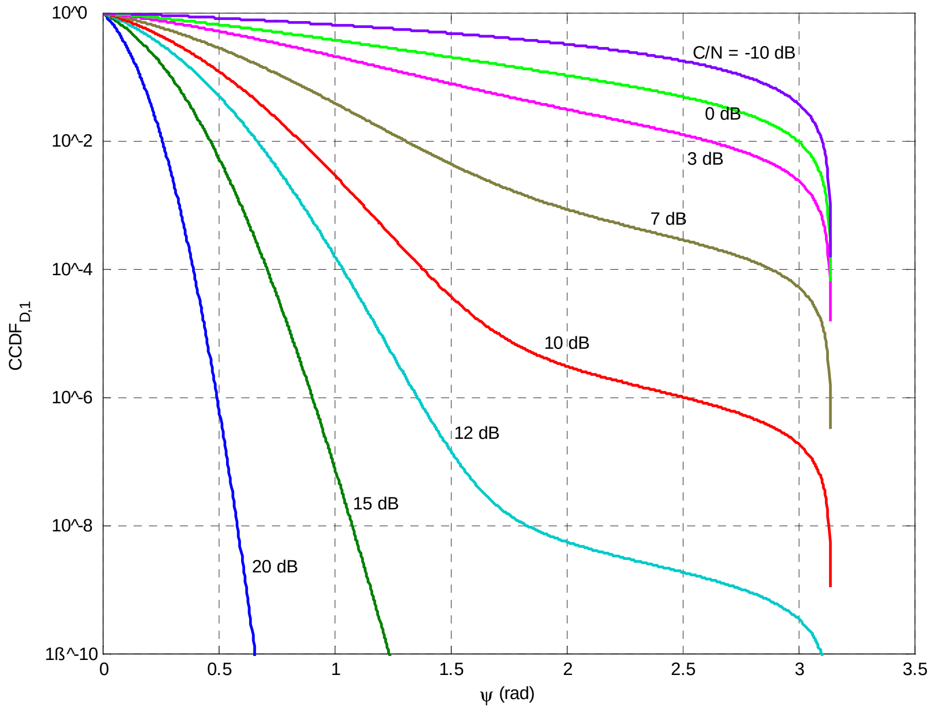

Fig 2-5b: CCDF of the Phase Difference of two Phasors Representing an Unmodulated Carrier plus Narrowband Gaussian Noise (Logarithmic Scale)

3. *M*-ary PSK Carrier plus Narrowband Gaussian Noise

In extension of section 2, the modulated carrier can be written as $y(t) = P\cos[\omega_c t + \psi + \zeta(t)]$ with the two independent random variables ψ [uniformly distributed in (0,2π)] and $\zeta(t)$, the modulation phase, which randomly takes any of the values $(2m\text{-}1)\pi/M$, $m \in (1, 2, ..., M)$ and $M = 2^{\mu}$ with $\mu \in \mathbf{N}$.

$$p_\zeta(\zeta_t) = 1/M \cdot \sum_{m=1}^{M} \delta(\zeta_t - \frac{2m-1}{M} \cdot \pi) .$$

The *M*-ary PSK carrier plus narrowband Gaussian noise is represented by

$$y(t) = P\cos[\omega_c t + \psi + \zeta(t)] + x(t) = X_c(t) \cdot \cos\omega_c t - X_s(t) \cdot \sin\omega_c t \tag{3-1}$$

or

$$\begin{aligned} y(t) &= V(t) \cdot \cos[\omega_c t + \psi + \phi(t)] \\ &= V(t) \cdot \cos[\psi + \phi(t)] \cdot \cos\omega_c t - V(t) \cdot \sin[\psi + \phi(t)] \cdot \sin\omega_c t \end{aligned} \tag{3-2}$$

with

$$X_c(t) = P\cos[\psi + \zeta(t)] + x_c(t)$$

and

$$X_s(t) = P\sin[\psi + \zeta(t)] + x_s(t) .$$

The random variables ψ, $\zeta(t)$, $x_c(t)$ and $x_s(t)$ are all statistically independent. Therefore the joint PDF is the product of the single-variable PDFs:

$$p_1(\psi, \zeta_t, x_{ct}, x_{st}) = \frac{1}{2\pi} \cdot \frac{1}{M} \sum_{m=1}^{M} \delta(\zeta_t - \frac{2m-1}{M} \cdot \pi) \cdot \frac{1}{2\pi} \exp(-\frac{x_{ct}^2 + x_{st}^2}{2\sigma_x^2}) \tag{3-3}$$

With the relations

$$x_c(t) = V(t)\cos[\psi + \phi(t)] - P\cos[\psi + \zeta(t)]$$

$$x_s(t) = V(t)\sin[\psi + \phi(t)] - P\sin[\psi + \zeta(t)]$$

the joint PDF $p_1(\psi,\zeta_t,x_{ct},x_{st})$ is transformed into the joint PDF $p_2(V_t,\phi_t,\psi,\zeta_t)$. The transformation determinant is

$$D = \begin{vmatrix} \partial\psi/\partial V_t & \partial\psi/\partial\phi_t & \partial\psi/\partial\psi & \partial\psi/\partial\zeta_t \\ \partial\zeta_t/\partial V_t & \partial\zeta_t/\partial\phi_t & \partial\zeta_t/\partial\psi & \partial\zeta_t/\partial\zeta_t \\ \partial x_{ct}/\partial\phi_t & \partial x_{ct}/\partial\phi_t & \partial x_{ct}/\partial\psi & \partial x_{ct}/\partial\zeta_t \\ \partial x_{st}/\partial V_t & \partial x_{st}/\partial\phi_t & \partial x_{st}/\partial\psi & \partial x_{st}/\partial\zeta_t \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \partial x_{ct}/\partial\phi_t & \partial x_{ct}/\partial\phi_t & \partial x_{ct}/\partial\psi & \partial x_{ct}/\partial\zeta_t \\ \partial x_{st}/\partial V_t & \partial x_{st}/\partial\phi_t & \partial x_{st}/\partial\psi & \partial x_{st}/\partial\zeta_t \end{vmatrix}$$

$$D = \begin{vmatrix} \partial x_{ct}/\partial\phi_t & \partial x_{ct}/\partial\phi_t \\ \partial x_{st}/\partial V_t & \partial x_{st}/\partial\phi_t \end{vmatrix} = \begin{vmatrix} \cos[\psi+\phi_t] & -V_t\sin[\psi+\phi_t] \\ \sin[\psi+\phi_t] & V_t\cos[\psi+\phi_t] \end{vmatrix} = V_t$$

Then we have for $V_t \geq 0$:

$$p_2(V_t,\phi_t,\psi,\zeta_t) = \frac{V_t}{2\pi}\cdot\frac{1}{M}\sum_{m=1}^{M}\delta(\zeta_t - \frac{2m-1}{M}\cdot\pi)\cdot\frac{1}{2\pi\sigma_x^{\ 2}}\cdot$$
$$\exp\left\{-\frac{1}{2\sigma_x^{\ 2}}[V_t\cos(\psi+\phi_t)-P\cos(\psi+\zeta_t)]^2+[V_t\sin(\psi+\phi_t)-P\sin(\psi+\zeta_t)]^2\right\}$$

$$p_2(V_t,\phi_t,\psi,\zeta_t) = \frac{V_t}{(2\pi)^2\cdot\sigma_x^{\ 2}}\cdot\exp\left\{-\frac{1}{2\sigma_x^{\ 2}}\left[V_t^{\ 2}+P^2-2V_tP\cos(\phi_t-\zeta_t)\right]\right\}\cdot$$
$$\frac{1}{M}\sum_{m=1}^{M}\delta(\zeta_t-\frac{2m-1}{M}\cdot\pi) \qquad (3\text{-}4)$$

Integrating equ. (3-4) with respect to ψ and ζ_t yields the joint PDF of V_t and ϕ_t:

$$p_3(V_t,\phi_t) = \int_0^{2\pi}\int_0^{2\pi} p_2(V_t,\phi_t,\psi,\zeta_t)\,d\psi\,d\zeta_t$$

$$= \frac{V_t}{2\pi\cdot\sigma_x^{\ 2}}\int_0^{2\pi}\exp\left\{-\frac{1}{2\sigma_x^{\ 2}}\left[V_t^{\ 2}+P^2-2V_tP\cos(\phi_t-\zeta_t)\right]\right\}\cdot\frac{1}{M}\sum_{m=1}^{M}\delta\left(\zeta_t-\frac{2m-1}{M}\cdot\pi\right)d\zeta_t$$

$$p_3(V_t,\phi_t) = \frac{V_t}{2\pi\cdot\sigma_x^{\ 2}}\cdot\exp\left[-\frac{V_t^{\ 2}+P^2}{2\sigma_x^{\ 2}}\right]\cdot\frac{1}{M}\sum_{m=1}^{M}\exp\left[\frac{V_tP}{\sigma_x^{\ 2}}\cos\left(\phi_t-\frac{2m-1}{M}\cdot\pi\right)\right]$$
$$= \frac{V_t}{2\pi\cdot M\sigma_x^{\ 2}}\cdot\sum_{m=1}^{M}\exp\left\{-\frac{P^2\sin^2\left(\phi_t-\frac{2m-1}{M}\cdot\pi\right)}{2\sigma_x^{\ 2}}\right\}\cdot\exp\left\{-\frac{\left[V_v-P\cos\left(\phi_t-\frac{2m-1}{M}\cdot\pi\right)\right]^2}{2\sigma_x^{\ 2}}\right\} \qquad (3\text{-}5)$$

where $V_t \geq 0$.

By comparison of the last form of $p_3(V_t,\phi_t)$ with equ. (2-5), it is seen that each of the M right-hand terms corresponds exactly to $1/M$ times the joint PDF of amplitude and phase of an un-modulated carrier plus narrowband Gaussian noise, but shifted in phase by $(2m\text{-}1)\pi/M$.

Thus, the joint PDF of amplitude and phase of an M-ary PSK modulated carrier plus narrow-band Gaussian noise is obtained as the M-fold superposition of the joint PDF of amplitude and phase of an unmodulated carrier plus Gaussian noise, the phase being staggered in steps of $2\pi/M$ and the amplitude weighted by $1/M$.

Because of the relation

$$\int_0^{2\pi}\exp\left[\frac{V_tP}{\sigma_x^{\ 2}}\cos\left(\phi_t-\frac{2m-1}{M}\cdot\pi\right)\right]d\phi_t = 2\pi\cdot I_0\left(\frac{V_tP}{\sigma_x^{\ 2}}\right) \qquad \text{for all } m\in(1, 2, ..., M)$$

the PDF of the amplitude of an M-ary PSK modulated carrier plus narrowband Gaussian noise is identical to that of an unmodulated carrier plus Gaussian noise (equ. (2-7)).

The PDF of the phase follows from integrating equ. (3-5) with respect to V_t:

$$p(\phi_t) = \frac{1}{2\pi \cdot M\sigma_x^2} \cdot \sum_{m=1}^{M} \exp\left\{-\frac{P^2}{2\sigma_x^2} \cdot \sin^2\left(\phi_t - \frac{2m-1}{M} \cdot \pi\right)\right\}$$

$$\cdot \int_0^\infty V_t \exp\left\{-\frac{1}{2\sigma_x^2} \cdot \left[V_v - P\cos\left(\phi_t - \frac{2m-1}{M} \cdot \pi\right)\right]^2\right\} dV_t$$

Using the notation $a = P/(\sqrt{2}\,\sigma_x)$ as in equ. (2-9) this becomes

$$p(\phi_t, a) = \frac{1}{2\pi}\exp(-a^2) + \frac{1}{2\pi \cdot M}\sqrt{\pi}\, a \sum_{m=1}^{M} \cos\left(\phi_t - \frac{2m-1}{M} \cdot \pi\right) \cdot \exp\left[-a^2 \cdot \sin^2\left(\phi_t - \frac{2m-1}{M} \cdot \pi\right)\right] \cdot \operatorname{erfc}\left[-a \cdot \cos\left(\phi_t - \frac{2m-1}{M} \cdot \pi\right)\right] \tag{3-6}$$

with $0 \le \phi_t \le 2\pi$.

The PDF of a 4-PSK modulated carrier plus narrowband Gaussian noise is illustrated in Fig. 3-1.

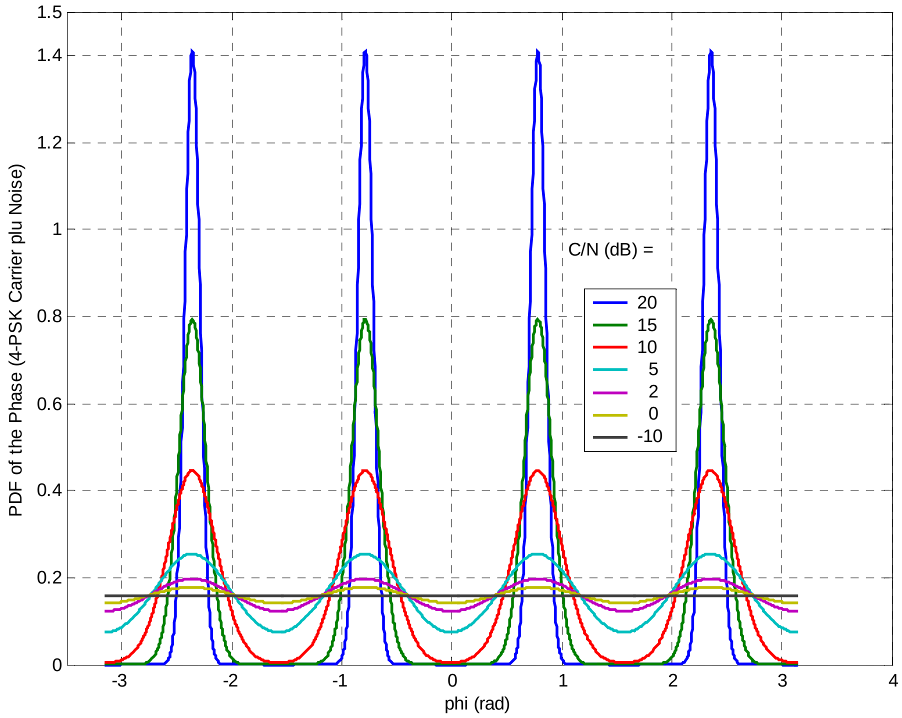

Fig. 3-1: PDF of the Phase of a 4-PSK Modulated Carrier plus Gaussian Noise

4. *M*-ary PSK Carrier plus Narrowband Gaussian Noise with Impact of Frequency Offset

When an *M*-ary PSK carrier is received by a non-coherent receiver, the demodulator will apply an estimated carrier frequency to the signal which, in general, does not coincide with the exact carrier frequency, but has some offset Δf. The *M*-ary PSK carrier and the additive Gaussian noise can then be represented by

$$y(t) = P\cos[\omega_c t + \Delta\omega t + \psi + \zeta(t)] + x(t) = X_c(t)\cdot\cos\omega_c t - X_s(t)\cdot\sin\omega_c t \qquad (4\text{-}1)$$

where $\Delta\omega = 2\pi\,\Delta f$. All other entries in equ. (4-1) have the same meaning as in equ. (3-1) above. Following the same procedure as in section 3, we obtain now

$$x_c(t) = V(t)\cos[\psi + \phi(t)] - P\cos[\psi + \zeta(t)]\cdot\cos\Delta\omega t + P\sin[\psi + \zeta(t)]\cdot\sin\Delta\omega t$$

and

$$x_s(t) = V(t)\sin[\psi + \phi(t)] - P\sin[\psi + \zeta(t)]\cdot\cos\Delta\omega t + P\cos[\psi + \zeta(t)]\cdot\sin\Delta\omega t\,.$$

The joint PDF of the variables ψ, $\zeta(t)$, $x_c(t)$ and $x_s(t)$ is again given by equ. (3-3), but now we have

$$x_{ct}^2 + x_{st}^2 = V_t^2 + P^2 - 2V_tP\cos\Delta\omega t\cdot\cos(\phi_t - \zeta_t) - 2V_tP\sin\Delta\omega t\cdot\sin(\phi_t - \zeta_t)$$

and thus

$$p_1(\psi,\zeta_t,x_{ct},x_{st}) = \frac{1}{2\pi}\cdot\frac{1}{M}\sum_{m=1}^{M}\delta\left(\zeta_t - \frac{2m-1}{M}\cdot\pi\right)\cdot\frac{1}{2\sigma_x^2}$$

$$\cdot\exp\left\{-\frac{1}{2\sigma_x^2}\left[V_t^2 + P^2 - 2V_tP\cos\Delta\omega t\cdot\cos(\phi_t - \zeta_t) - 2V_tP\sin\Delta\omega t\cdot\sin(\phi_t - \zeta_t)\right]\right\}.$$

The joint PDF of the variables V_t, ϕ_t, ψ, ζ_t then is

$$p_2(V_t,\phi_t,\psi,\zeta_t) = \frac{V_t}{(2\pi)^2\sigma_x^2}\cdot\exp\left\{-\frac{1}{2\sigma_x^2}\left[V_t^2 + P^2 - 2V_tP\cos(\phi_t - \zeta_t - \Delta\omega t)\right]\right\}$$
$$\cdot\frac{1}{M}\sum_{m=1}^{M}\delta\left(\zeta_t - \frac{2m-1}{M}\cdot\pi\right) \qquad (4\text{-}2)$$

The joint amplitude and phase PDF is obtained from integration with respect to ψ and ζ_t:

$$p_3(V_t,\phi_t) = \frac{V_t}{2\pi\cdot\sigma_x^2}\cdot\exp\left[-\frac{V_t^2 + P^2}{2\sigma_x^2}\right]\cdot\frac{1}{M}\sum_{m=1}^{M}\exp\left[\frac{V_tP}{\sigma_x^2}\cos\left(\phi_t - \frac{2m-1}{M}\cdot\pi - \Delta\omega t\right)\right]$$

$$p_3(V_t,\phi_t) = \frac{V_t}{2\pi \cdot M\sigma_x^{\;2}} \cdot \sum_{m=1}^{M} \exp\left\{-\frac{P^2}{2\sigma_x^{\;2}}\sin^2\left(\phi_t - \frac{2m-1}{M}\cdot\pi - \Delta\omega t\right)\right\}\cdot$$
$$\cdot\exp\left\{-\frac{1}{2\sigma_x^{\;2}}\left[V_t - P\cos\left(\phi_t - \frac{2m-1}{M}\cdot\pi - \Delta\omega t\right)\right]^2\right\} \qquad (4\text{-}3)$$

Integration of equ. (4-3) with respect to ϕ_t leads to the amplitude PDF with consideration of Frequency offset. The result is the same as given in equ. (2-7) for an unmodulated carrier plus narrowband Gaussian noise .

The phase PDF is obtained by integrating equ. (4-3) with respect to V_t. The integration follows the same method as used in equ. (2-8) and equ. (3-5). Due to the frequency offset, the phase PDF is now also dependent on time t. The result is:

$$p(\phi,t) = \frac{1}{2\pi}\exp\left[-\frac{P^2}{2\sigma_x^{\;2}}\right] + \frac{1}{2\pi\cdot M}\sqrt{\frac{\pi}{2}}\,\frac{P}{\sigma_x}\cdot\sum_{m=1}^{M}\cos\left(\phi_t - \frac{2m-1}{M}\cdot\pi - \Delta\omega t\right)$$
$$\cdot\exp\left[-\frac{P^2}{2\sigma_x^{\;2}}\cdot\sin^2\left(\phi_t - \frac{2m-1}{M}\cdot\pi - \Delta\omega t\right)\right]\cdot \mathrm{erfc}\left[-\frac{P}{\sqrt{2}\sigma_x}\cdot\cos\left(\phi_t - \frac{2m-1}{M}\cdot\pi - \Delta\omega t\right)\right] \qquad (4\text{-}4)$$

Some representative cases are illustrated in Figures 4-1 (a) through (c).

Fig. 4-1: Probability Density Function of the Phase of a 4-PSK Carrier plus NB Gaussian Noise with Impact of Frequency Offset

(a) C/N = 10 dB and Δf = 20 Hz

(b) C/N = 10 dB and Δf = 100 Hz

(c) C/N = 2 dB and Δf = 100 Hz

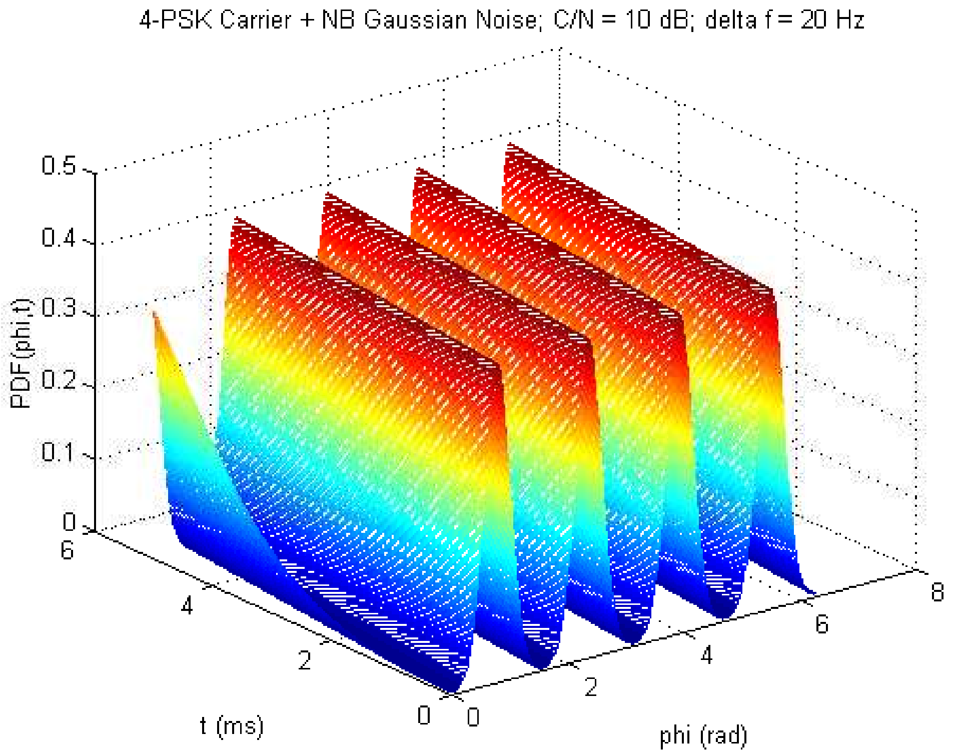

Fig. 4-1 (a)

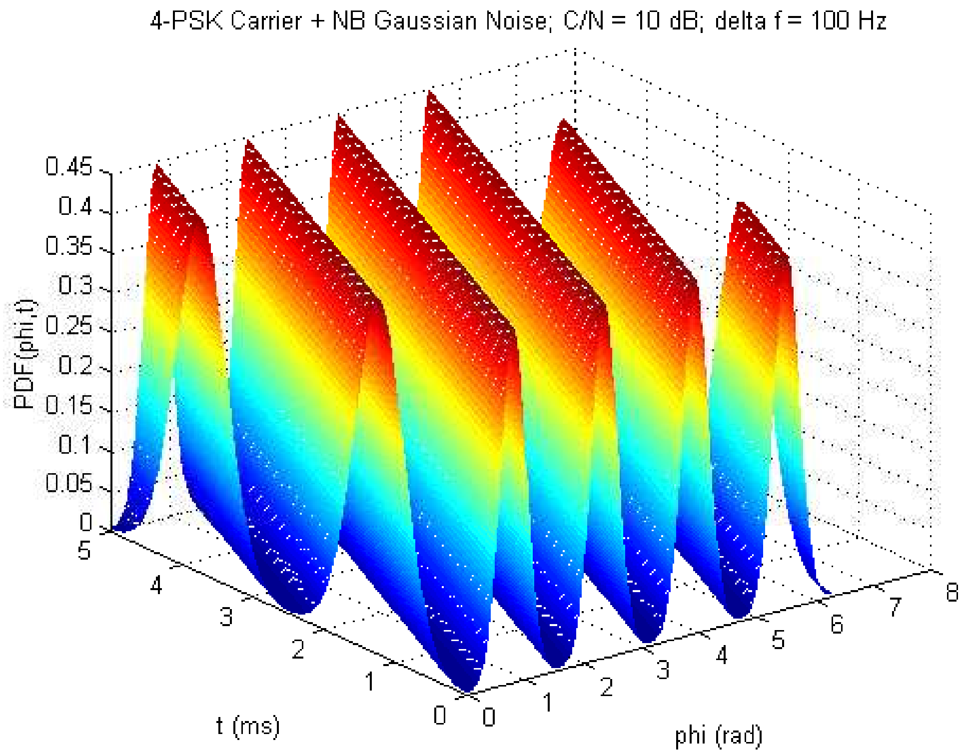

Fig. 4-1 (b)

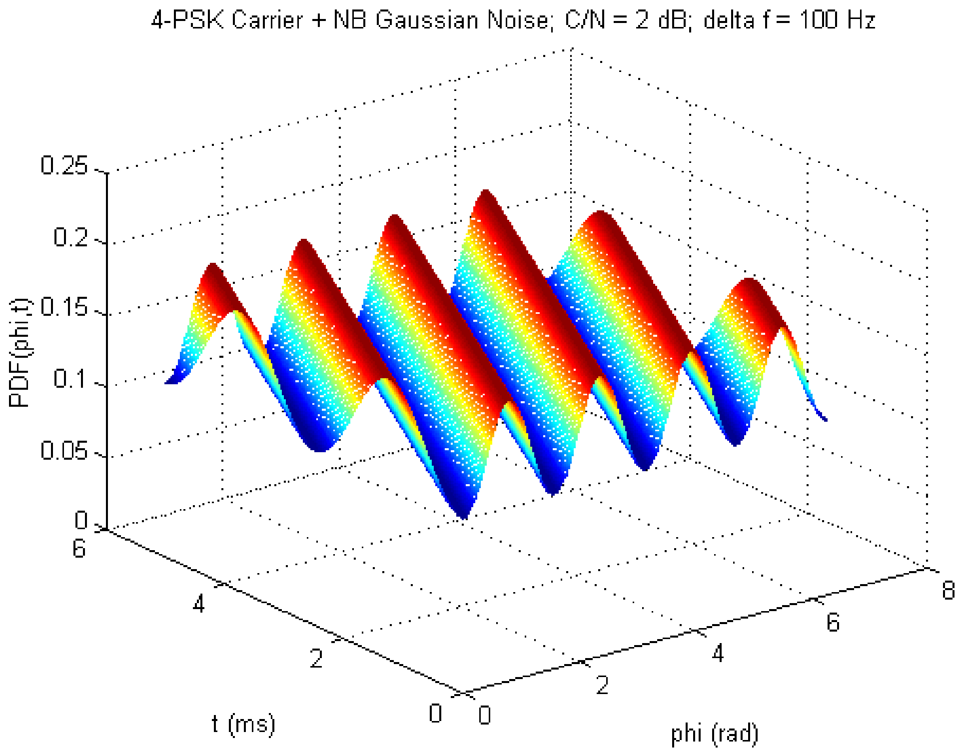

Fig. 4-1 (c)

5. Unmodulated Carrier plus Narrowband Gaussian Noise with Impact of Frequency Offset

The phase PDF formula according to equ. (4-4) - although derived for M-ary PSK carriers with additive narrowband Gaussian noise - can be used also to compute the PDF of the phase of an unmodulated carrier plus narrowband Gaussian noise by formally setting $M = 1$ and restricting m to 1. This case is of interest when considering the topic of carrier synchronisation in communication receivers. An example is shown in Fig. 5-1.

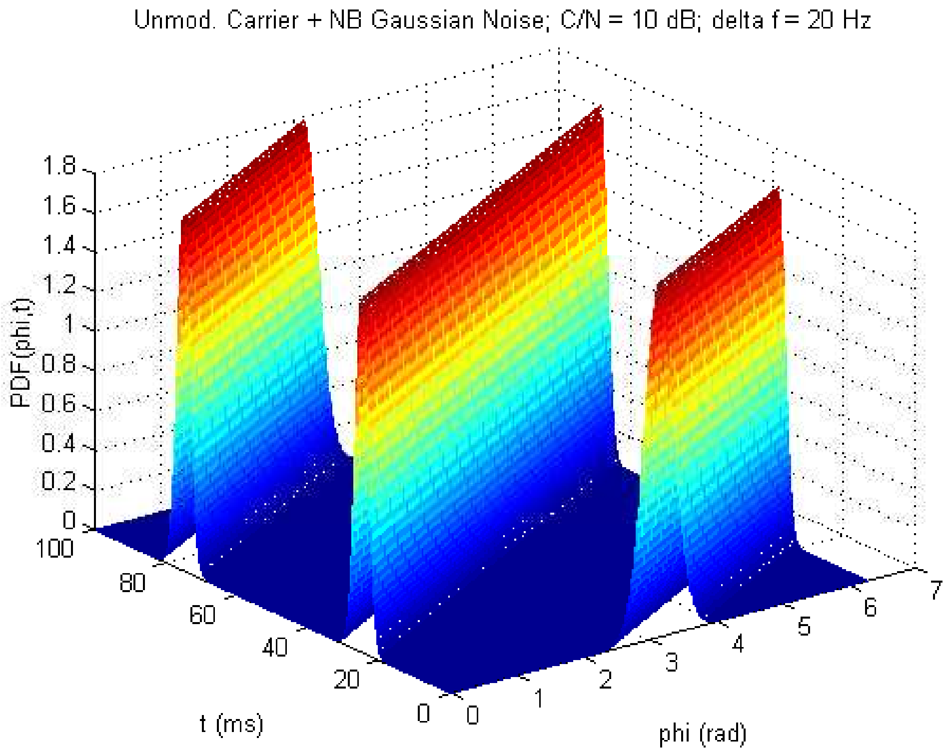

Fig. 5-1: Probability Density Function of the Phase of an Unmodulated Carrier plus Narrowband Gaussian Noise with Impact of Frequency Offset

6. References

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